

STRONG GENERAL POSITION

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ABSTRACT. We say that a finite set S of points in \mathbb{R}^d is in **strong general position** if for any collection $\{F_1, \dots, F_r\}$ of r pairwise disjoint subsets of S ($1 \leq r \leq |S|$) we have: $d - \dim \bigcap_{\nu=1}^r \text{aff } F_\nu = \min(d + 1, \sum_{\nu=1}^r (d - \dim \text{aff } F_\nu))$. In this paper we reduce the set of conditions that one has to check in order to determine if S is in **strong general position**.

1. INTRODUCTION

A set S of points in \mathbb{R}^d is said to be in **general position** if every set of $d + 1$ or fewer points of S is affinely independent, or, in other words, if any k -flat in \mathbb{R}^d ($1 \leq k \leq d - 1$) contains at most $k + 1$ points of S . This condition can be simplified:

$S \subset \mathbb{R}^d$ is in general position if either $|S| \leq d + 1$ and S is affinely independent, or $|S| \geq d + 1$ and every $d + 1$ points of S are affinely independent.

General position is a rather weak property. E.g., the vertices of a regular $2m$ -gon $P \subset \mathbb{R}^2$ are in general position, even though the main diagonals of P cross in a common point, and, moreover, opposite edges span parallel lines.

In this paper we consider a stronger property, called **strong general position** (SGP). A finite set $S \subset \mathbb{R}^d$ is said to be in SGP if, for any collection $\{F_1, \dots, F_r\}$ of r pairwise disjoint subsets of S ($1 \leq r \leq |S|$), the affine hulls $\text{aff } F_1, \dots, \text{aff } F_r$ intersect as if they were flats chosen at random. The formal condition is :

$$(1.1) \quad \dim \bigcap_{\nu=1}^r \text{aff } F_\nu = \max(-1, d - \sum_{\nu=1}^r (d - \dim \text{aff } F_\nu)).$$

As we shall see, this condition implies (ordinary) general position.

If we want to check whether a given large set $S \subset \mathbb{R}^d$ is in SGP, we are faced with a huge number of conditions of the form (1.1). The purpose of this paper is to reduce this number, i.e., to find a much smaller, essentially minimal set of conditions that will ensure that a given finite set $S \subset \mathbb{R}^d$ is in SGP. This reduction could be of use to anyone who wishes to work seriously with the notion of SGP. We use it in [PS] to show that points chosen on the

moment curve M^d in \mathbb{R}^d ($M^d = \{(t, t^2, \dots, t^d) : t \in \mathbb{R}\}$) are "usually" in SGP.

The notion of strong general position has been used (under the name "strong independence") by Reay in [R] and by Doignon and Valette in [DV].

Strong general position plays an important role in connection with Tverberg's Theorem:

Theorem 1.1. (*H. Tverberg, 1966*) *Let a_1, \dots, a_n be points in \mathbb{R}^d . If $n > (d+1)(r-1)$, then the set $N = \{1, \dots, n\}$ of indices can be partitioned into r disjoint parts N_1, \dots, N_r in such a way that the r convex hulls $\text{conv}\{a_i : i \in N_j\}$ ($j = 1, \dots, r$) have a point in common.*

(This formulation covers also the case where the points a_1, \dots, a_n are not all distinct.) The original proof (see [T66]) was quite difficult. In 1981 Tverberg published another proof, much simpler than the original one (see [T81]). Sarkaria [Sa] gave a quite accessible proof, with some algebraic flavor. It seems that the simplest proof so far is due to Roudneff [Ro]. See [M] §8.3 for further information.

The numbers $T(d, r) = (d+1)(r-1) + 1$ are known as Tverberg numbers. The condition $n \geq T(d, r)$ in Tverberg's theorem is extremely tight. If $n < T(d, r)$, and the points a_1, \dots, a_n are in SGP, then for any r -partition N_1, \dots, N_r of the set $N = \{1, \dots, n\}$, even the intersection of the **affine** hulls $\text{aff}\{a_i : i \in N_j\}$ ($j = 1, \dots, r$) is empty. (See details in the next section.)

Our reduction (see Theorem 2.2 below) will show that if $S \subset \mathbb{R}^d$ is a finite set in general position, then S is in SGP iff, for any collection $\{F_1, \dots, F_r\}$ of pairwise disjoint non-empty subsets of S (with $2 \leq r \leq d+1$, $|F_\nu| \leq d$ for all i) of total size $m = \sum_{\nu=1}^r |F_\nu|$, the intersection $\cap_{i=1}^r \text{aff } F_\nu$ is a single point if $m = T(d, r)$, or empty if $m < T(d, r)$.

As we shall see in Section 3, finite subsets of \mathbb{R}^d are "usually" in SGP, in the following strong sense: Given d and n , there exists a polynomial $P(= P_{d,n})$, not identically zero, in nd scalar variables: $P(\vec{x}_1, \dots, \vec{x}_n) = P(x_{11}, \dots, x_{1d}, \dots, x_{n1}, \dots, x_{nd})$, such that any n points $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^d$ are (distinct and) in SGP unless $P(\vec{a}_1, \dots, \vec{a}_n) = 0$.

There are notions of independence that are even stronger than SGP. In fact, the first proof of Tverberg's Theorem in [T66] runs under the assumption that the points $a_1, \dots, a_n \in \mathbb{R}^d$ are **algebraically independent**, i.e., that the nd coordinates a_{ij} ($1 \leq i \leq n, 1 \leq j \leq d$) are algebraically independent over the field of rational numbers. A limiting argument then establishes Tverberg's Theorem for all $a_1, \dots, a_n \in \mathbb{R}^d$.

2. STRONG GENERAL POSITION

A $(d - k)$ -dimensional flat in \mathbb{R}^d ($0 \leq k \leq d$) is the set of solutions of a system of k linearly independent linear equations (not necessarily homogeneous) in d variables. It follows that if A_1, \dots, A_r are flats in \mathbb{R}^d , and $\dim A_\nu = d - k_\nu$ for $\nu = 1, \dots, r$, then the intersection $\bigcap_{\nu=1}^r A_\nu$ will "usually" be a flat of dimension $d - \sum_{\nu=1}^r k_\nu$ (when $\sum_{\nu=1}^r k_\nu \leq d$), or \emptyset (when $\sum_{\nu=1}^r k_\nu > d$). The dimension of the empty set \emptyset is, by definition, -1 . We **always** have:

either $\dim \bigcap_{\nu=1}^r A_\nu \geq d - \sum_{\nu=1}^r k_\nu$, **or** $\bigcap_{\nu=1}^r A_\nu = \emptyset$.

In view of these observations we define:

Definition 2.1. A finite set $S \subset \mathbb{R}^d$ is in **strong general position (SGP)** if:

(a) S is in general position, i.e., every subset of S of size $\leq d + 1$ is affinely independent or, equivalently, $\dim \text{aff } F = \min(d, |F| - 1)$ for all subsets $F \subseteq S$.

(b) For any collection $\{F_1, \dots, F_r\}$ of r pairwise disjoint subsets of S ($1 \leq r \leq |S|$):

$$(2.1) \quad d - \dim \bigcap_{\nu=1}^r \text{aff } F_\nu = \min(d + 1, \sum_{\nu=1}^r (d - \dim \text{aff } F_\nu)).$$

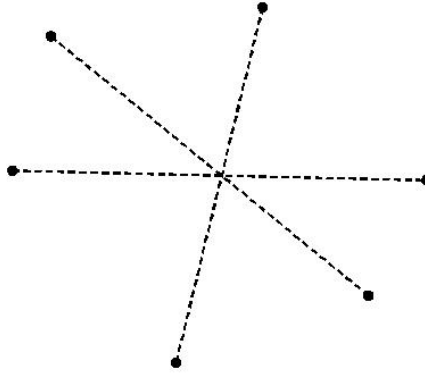


FIGURE 1. These six points are in general position but not in SGP

Remark: Condition (a) in the definition above follows from condition (b). In fact, if S is not in general position, i.e., if S has an affinely dependent subset of size $\leq d + 1$, consider a minimal affinely dependent subset F of S . Assume $|F| = k$, $3 \leq k \leq d + 1$. Then $\dim \text{aff } F = k - 2$. The set F admits

a Radon partition $F = A \cup B$, $A \cap B = \emptyset$, $\text{conv } A \cap \text{conv } B \neq \emptyset$. Assume $|A| = a(< k)$, $|B| = k - a(< k)$. The sets A, B are affinely independent. Thus $\text{aff } A \cap \text{aff } B \neq \emptyset$, even though

$$\begin{aligned} & d - \dim \text{aff } A + d - \dim \text{aff } B \\ &= d - (a - 1) + d - (k - a - 1) \\ &= 2d - k + 2 \geq 2d - (d + 1) + 2 = d + 1. \end{aligned}$$

Our next aim is to show that if S is a finite subset of \mathbb{R}^d in general position, then we have to check only a small fraction of the conditions listed in (b) above in order to determine whether S is in SGP. We shall do this in five steps. The final result is stated as Theorem 2.2 below.

(A) Suppose $S \subset \mathbb{R}^d$ is finite and in general position. Then S is in SGP iff (2.1) holds for any collection F_1, \dots, F_r of pairwise disjoint subsets of S that satisfy $1 \leq |F_\nu| \leq d$ for $\nu = 1, 2, \dots, r$.

Proof. If $F_\nu = \emptyset$ for some $1 \leq \nu \leq r$, then (2.1) holds automatically. (Both sides of the equality are $d + 1$.)

If $|F_\mu| \geq d + 1$ for some $1 \leq \mu \leq r$, then $\text{aff } F_\mu = \mathbb{R}^d$, and removing F_μ from the collection does not affect the intersection $\bigcap_{\nu=1}^r \text{aff } F_\nu$ on the left-hand side, nor the sum $\sum_{\nu=1}^r (d - \dim \text{aff } F_\nu)$ on the right-hand side. Thus (2.1) holds for the given collection $\{F_1, \dots, F_r\}$ iff it holds for the subcollection obtained by removing all F_ν 's with $|F_\nu| > d$. (If all $|F_\nu|$'s are $> d$, then both sides of (2.1) are 0.) \square

(B) If $S \subset \mathbb{R}^d$ is finite and in general position, and F_1, \dots, F_r are pairwise disjoint subsets of S that satisfy $1 \leq |F_\nu| \leq d$ for all ν and $\sum_{\nu=1}^r |F_\nu| = m$, then the equality (2.1) is equivalent to the condition:

$$(2.2) \quad \begin{cases} \dim \bigcap_{\nu=1}^r \text{aff } F_\nu = m - T(d, r) & \text{if } m \geq T(d, r) \\ \bigcap_{\nu=1}^r \text{aff } F_\nu = \emptyset & \text{if } m < T(d, r). \end{cases}$$

Proof. (2.1) is equivalent to:

$$\begin{aligned}
\dim \bigcap_{\nu=1}^r \text{aff } F_\nu &= d - \min(d+1, \sum_{\nu=1}^r (d - \dim \text{aff } F_\nu)) \\
&= d - \min(d+1, rd + r - \sum_{\nu=1}^r |F_\nu|) \\
&= \max(-1, m - r(d+1) + d) \\
&= \max(-1, m - T(d, r)).
\end{aligned}$$

□

(C) Suppose $S \subset \mathbb{R}^d$ is finite and in general position. Then S is in SGP iff: for any r pairwise disjoint subsets F_1, \dots, F_r of S with $1 \leq |F_\nu| \leq d$ for all ν ($2 \leq r \leq |S|$) and $\sum_{\nu=1}^r |F_\nu| = m$:

$$(2.3) \quad \left| \bigcap_{\nu=1}^r \text{aff } F_\nu \right| = \begin{cases} 1 & \text{if } m = T(d, r) \\ 0 & \text{if } m < T(d, r). \end{cases}$$

(Note that (2.3) always holds for $r = 1$: when $r = 1$, $m = |F_1|$, $T(d, 1) = 1$ and $\dim \text{aff } F_1 = |F_1| - 1 = m - T(d, 1)$.)

Proof. The "only if" direction is clear: condition (2.3) is the restriction of condition (2.2) to the case $m \leq T(d, r)$.

For the "if" direction: assume (2.2) fails for some $m > T(d, r)$, i.e., there are some r pairwise disjoint subsets F_1, \dots, F_r of S , $1 \leq |F_\nu| \leq d$ for all ν , such that $m = \sum_{\nu=1}^r |F_\nu| > T(d, r)$, and $\dim \bigcap_{\nu=1}^r \text{aff } F_\nu \neq m - T(d, r)$.

Note that this can happen only for $r \geq 2$.

Among all the "violations" (F_1, \dots, F_r) of (2.2) with $m (= \sum_{\nu=1}^r |F_\nu|) > T(d, r)$ (where r is not fixed in advance), choose one with m as small as possible. Then one of the following holds:

Case I: $\bigcap_{\nu=1}^r \text{aff } F_\nu = \emptyset$.

Case II: $\dim \bigcap_{\nu=1}^r \text{aff } F_\nu > m - T(d, r)$.

In case I, choose nonempty subsets $G_\nu \subseteq F_\nu$ such that $\sum_{\nu=1}^r |G_\nu| = T(d, r)$. This is possible, since $T(d, r) = (d+1)(r-1) + 1 \geq r$. The sets G_1, \dots, G_r violate condition (2.3), since $\bigcap_{\nu=1}^r \text{aff } G_\nu \subset \bigcap_{\nu=1}^r \text{aff } F_\nu = \emptyset$, even though $\sum_{\nu=1}^r |G_\nu| = T(d, r)$.

In case II, choose an index μ such that $|F_\mu| > 1$. Pick a point $p \in F_\mu$ and define $G_\mu = F_\mu \setminus \{p\}$, $G_\nu = F_\nu$ for all $\nu \neq \mu$. Then $\dim \text{aff } G_\mu = \dim \text{aff } F_\mu - 1$, hence $\text{aff } G_\mu = H \cap \text{aff } F_\mu$ for some hyperplane $H \subset \mathbb{R}^d$. Therefore, $\bigcap_{\nu=1}^r \text{aff } G_\nu = H \cap \bigcap_{\nu=1}^r \text{aff } F_\nu$. This implies:

either

$$(2.4) \quad \bigcap_{\nu=1}^r \text{aff } G_\nu = \emptyset,$$

or

$$(2.5) \quad \dim \bigcap_{\nu=1}^r \text{aff } G_\nu \geq -1 + \dim \bigcap_{\nu=1}^r \text{aff } F_\nu > m - 1 - T(d, r),$$

where $m - 1 = \sum_{\nu=1}^r |G_\nu|$.

If (2.4) holds, then we have a smaller violation of (2.2) if $m - 1 > T(d, r)$ (contrary to our choice of (F_1, \dots, F_r)), or a violation of (2.3), if $m - 1 = T(d, r)$.

If (2.5) holds, then again we have a smaller violation of (2.2) if $m - 1 > T(d, r)$, or a violation of (2.3) if $m - 1 = T(d, r)$. □

In the next step we discard conditions that relate to the case $m < T(d, r)$ and are not minimal.

If F_1, \dots, F_r are pairwise disjoint subsets of S , and $|F_\nu| = d + 1 - \varepsilon_\nu$, for $\nu = 1, \dots, r$, then $m = \sum_{\nu=1}^r |F_\nu| = r(d + 1) - \sum_{\nu=1}^r \varepsilon_\nu$, whereas $T(d, r) = r(d + 1) - d$. Thus $m < T(d, r)$ iff $\sum_{\nu=1}^r \varepsilon_\nu > d$. If, for some proper subset R' of $\{1, \dots, r\}$ of size r' , we have $\sum_{\nu \in R'} \varepsilon_\nu > d$, then $m' = \sum_{\nu \in R'} |F_\nu| = r'(d + 1) - \sum_{\nu \in R'} \varepsilon_\nu < r'(d + 1) - d = T(d, r')$. If $\bigcap_{\nu \in R'} \text{aff } F_\nu = \emptyset$, then, a fortiori, $\bigcap_{\nu=1}^r \text{aff } F_\nu = \emptyset$.

This reduces the criterion for SGP to the following:

(D) Suppose $S \subset \mathbb{R}^d$ is finite and in general position, $|S| > d + 1$. Then S is in SGP iff:

For any r pairwise disjoint subsets F_1, \dots, F_r of S ($2 \leq r \leq d + 1$), if $|F_\nu| = d + 1 - \varepsilon_\nu$, $1 \leq \varepsilon_\nu \leq d$ for $\nu = 1, \dots, r$, then $\bigcap_{\nu=1}^r \text{aff } F_\nu$ is a singleton if $\sum_{\nu=1}^r \varepsilon_\nu = d$, and empty if $d < \sum_{\nu=1}^r \varepsilon_\nu \leq d + \min\{\varepsilon_\nu, 1 \leq \nu \leq r\}$.

Now comes the final reduction:

Theorem 2.2. *Suppose $S \subset \mathbb{R}^d$ is in general position and $|S| > d + 1$. Then S is in SGP iff:*

For any r pairwise disjoint subsets F_1, \dots, F_r of S ($2 \leq r \leq d + 1$), if $|F_\nu| = d + 1 - \varepsilon_\nu$, $1 \leq \varepsilon_\nu \leq d$ for $\nu = 1, \dots, r$, then $\bigcap_{\nu=1}^r \text{aff } F_\nu$ is a singleton if $\sum_{\nu=1}^r \varepsilon_\nu = d$, and empty if

either $\sum_{\nu=1}^r \varepsilon_\nu = d + 1$,

or $|S| = r(d + 1) - \sum_{\nu=1}^r \varepsilon_\nu$, and $d + 1 < \sum_{\nu=1}^r \varepsilon_\nu \leq d + \min\{\varepsilon_\nu, 1 \leq \nu \leq r\}$.

Remarks: (a) $|S| = r(d+1) - \sum_{\nu=1}^r \varepsilon_\nu$ means just that $F_1 \cup \dots \cup F_r = S$.
 (b) If $|S| \geq d(d+1)$, then we can dispense with the second clause in Theorem 2.2, and the condition becomes:

$$(2.6) \quad \left| \bigcap_{\nu=1}^r \text{aff } F_\nu \right| = \begin{cases} 1 & \text{if } \sum_{\nu=1}^r \varepsilon_\nu = d \\ 0 & \text{if } \sum_{\nu=1}^r \varepsilon_\nu = d+1. \end{cases}$$

Proof. The "only if" direction is clear: the conditions in Theorems 2.2 are just a subset of the conditions in (D) above.

As for the "if" direction:

Assume one of the conditions in (D) that is missing in Theorem 2.2 is violated: F_1, \dots, F_r are r pairwise disjoint subsets of S , ($2 \leq r \leq d+1$), $|F_\nu| = d+1 - \varepsilon_\nu$, $1 \leq \varepsilon_\nu \leq d$ for $\nu = 1, \dots, r$, $d+1 < \sum_{\nu=1}^r \varepsilon_\nu \leq d + \min\{\varepsilon_\nu, 1 \leq \nu \leq r\}$, $|S| > r(d+1) - \sum_{\nu=1}^r \varepsilon_\nu$ (i.e. $S \supsetneq \cup_{\nu=1}^r F_\nu$) and still $\cap_{\nu=1}^r \text{aff } F_\nu \neq \emptyset$.

Choose such a violation with $\sum_{\nu=1}^r \varepsilon_\nu$ as small as possible. Note that $\sum_{\nu=1}^r \varepsilon_\nu \geq d+2$. Choose an index μ , $1 \leq \mu \leq r$, with $\varepsilon_\mu \geq 2$, and a point $q \in S \setminus \cup_{\nu=1}^r F_\nu$, replace F_μ by $F'_\mu = F_\mu \cup \{q\}$, and define $F'_\nu = F_\nu$ for all $\nu \neq \mu$. Now $|F'_\nu| = d+1 - \varepsilon'_\nu$, where $\varepsilon'_\nu = \varepsilon_\nu$ for $\nu \neq \mu$, and $\varepsilon'_\mu = \varepsilon_\mu - 1$. Clearly $\cap_{\nu=1}^r \text{aff } F'_\nu \supset \cap_{\nu=1}^r \text{aff } F_\nu \neq \emptyset$. If $\sum_{\nu=1}^r \varepsilon'_\nu > d+1$ and $|S| > r(d+1) - \sum_{\nu=1}^r \varepsilon'_\nu$, then we have a violation of a condition in (D) that is missing in Theorem 2.2 with $\sum_{\nu=1}^r \varepsilon'_\nu < \sum_{\nu=1}^r \varepsilon_\nu$, contrary to our earlier choice. If $\sum_{\nu=1}^r \varepsilon'_\nu = d+1$, or $|S| = r(d+1) - \sum_{\nu=1}^r \varepsilon'_\nu$, then we have a violation of one of the conditions in Theorem 2.2. \square

Conclusion: The following "recipe" states explicitly what has to be checked in order to ascertain that a given list (a_1, \dots, a_n) of points in \mathbb{R}^d consists of n distinct points in SGP. In order to learn how to perform the various checks, the reader is advised to consult Section 3 below.

Step I: Check that the given points a_1, \dots, a_n are (distinct and) in (ordinary) general position.

Step II: Consider collections $\mathcal{F} = \{F_1, \dots, F_r\}$ of pairwise disjoint subsets of $\{a_1, \dots, a_n\}$. Assume $1 \leq |F_\nu| \leq d$ for all $1 \leq \nu \leq r$, say $|F_\nu| = d+1 - \varepsilon_\nu$, where $1 \leq \varepsilon_\nu \leq d$. (To avoid duplication, you may assume that $\min\{i : a_i \in F_\nu\} < \min\{i : a_i \in F_{\nu+1}\}$ for $\nu = 1, 2, \dots, r-1$.) Denote by m the total size $\sum_{\nu=1}^r |F_\nu|$ of \mathcal{F} ($m = r(d+1) - \sum_{\nu=1}^r \varepsilon_\nu$).

(A) If $m = T(d, r)$ (i.e., $\sum_{\nu=1}^r \varepsilon_\nu = d$), check that $|\cap_{\nu=1}^r \text{aff } F_\nu| = 1$. This should be done for all r , $2 \leq r \leq \min\{d, \lfloor \frac{n+d}{d+1} \rfloor\}$.

(B) If $m = T(d, r) - 1$ (i.e., $\sum_{\nu=1}^r \varepsilon_\nu = d + 1$), check that $\cap_{\nu=1}^r \text{aff } F_\nu = \emptyset$. This should be done for all r , $3 \leq r \leq \min\{d + 1, \lceil \frac{n+d+1}{d+1} \rceil\}$.

(C) Define $\varepsilon_0 = \min\{\varepsilon_1, \dots, \varepsilon_d\}$. If $n = m \leq T(d, r) - 2$ (i.e., $\sum_{\nu=1}^r \varepsilon_\nu \geq d + 2$), but $\sum_{\nu=1}^r \varepsilon_\nu - \varepsilon_0 \leq d$ (which implies $\varepsilon_0 \geq 2$), check that $\cap_{\nu=1}^r \text{aff } F_\nu = \emptyset$. This should be done only for $3 \leq r = \lceil \frac{n+d+2}{d+1} \rceil \leq \lceil \frac{d+2}{2} \rceil$. In fact, clause (C) is applicable iff $d \geq 4$, $3 \leq r \leq \lceil \frac{d+2}{2} \rceil$ and $T(d, r) - \lceil \frac{d}{r-1} \rceil \leq n \leq T(d, r) - 2$.

3. POINTS ARE "USUALLY" IN SGP

Let $X = (\vec{x}_1, \dots, \vec{x}_t)$ be a sequence of t points in \mathbb{R}^d . Denote by x_{k1}, \dots, x_{kd} the coordinates of \vec{x}_k ($k = 1, 2, \dots, t$). We regard the td quantities $x_{k\nu}$ ($1 \leq k \leq t, 1 \leq \nu \leq d$) as real variables, and propose to find a non-zero polynomial $P = P_{t,d}$ in these variables, in such a way that the points $\vec{x}_1, \dots, \vec{x}_t$ are (distinct and) in SGP, unless $P(\vec{x}_1, \dots, \vec{x}_t) = 0$. As we have seen in the preceding section, strong general position is a conjunction of a long list of conditions. For each condition (E) on the list we shall produce a non-zero polynomial P_E , such that the violation of condition (E) by the points $\vec{x}_1, \dots, \vec{x}_t$ will imply $P_E(\vec{x}_1, \dots, \vec{x}_t) = 0$. The polynomial $P_{t,d}$ promised above will be the product of all these polynomials P_E .

Denote by $M(X)$ the $(d+1) \times t$ matrix whose k -th column consists of the number 1, followed by the coordinates of \vec{x}_k , i.e., $\begin{pmatrix} 1 \\ \vec{x}_k \end{pmatrix} = (1, x_{k1}, \dots, x_{kd})^t$. For a subsequence B of X of length b we denote by $M(B)$ the $(d+1) \times b$ submatrix of $M(X)$ that consists of the columns that correspond to points of B only.

Let us start with the condition that the points of X be distinct and in (ordinary) general position. If $t = d + 1$, this means that the points $\vec{x}_1, \dots, \vec{x}_{d+1}$ are affinely independent, i.e., that $\det M(X) \neq 0$, so the corresponding polynomial is just $\det M(X)$. If $t > d + 1$, this means that each $d + 1$ of the points $\vec{x}_1, \dots, \vec{x}_t$ are affinely independent, so the corresponding polynomial is the product of the determinants of all $\binom{t}{d+1}$ $(d+1) \times (d+1)$ square submatrices of $M(X)$. If $t < d + 1$, then general position of the points of X is the same as affine independence, so the condition is: $\text{rank } M(X) = t$. This means that $M(X)$ has at least one $t \times t$ square non-singular submatrix, and the corresponding polynomial is the sum of squares of the determinants of all $\binom{d+1}{t}$ $t \times t$ square submatrices of $M(X)$.

When $t \leq d + 1$, SGP is the same as (ordinary) general position, so we can stop here. Assume, from now on, that $t > d + 1$. We assume that the points $\vec{x}_1, \dots, \vec{x}_t$ are (distinct and) in general position, (otherwise, some polynomial

we found already vanishes at $\vec{x}_1, \dots, \vec{x}_t$, and proceed with the additional conditions, as they appear in (D) in Section 2 above. (To be precise, we use the notation of (D) ($|F_\nu| = d + 1 - \varepsilon_\nu$, $1 \leq \varepsilon_\nu \leq d$ for $\nu = 1, \dots, r$), but we do not use the reduction from (C) to (D), except for the fact that $2 \leq r \leq d + 1$.)

Let F_1, \dots, F_r ($2 \leq r \leq d + 1$) be disjoint subsets of $\vec{x}_1, \dots, \vec{x}_t$, $|F_\nu| = d + 1 - \varepsilon_\nu$, $1 \leq \varepsilon_\nu < d$ for $\nu = 1, \dots, r$.

Case I: If $\sum_{\nu=1}^r \varepsilon_\nu = d$, then $|\cap_{\nu=1}^r \text{aff } F_\nu| = 1$. Denote by \vec{z} the unique point of $\cap_{\nu=1}^r \text{aff } F_\nu$. For each ν , $1 \leq \nu \leq r$, \vec{z} can be expressed as an affine combination (i.e., a linear combination with sum of coefficients 1) of the points of F_ν . This expression is unique since F_ν is affinely independent. Thus $(\frac{1}{\vec{z}}) = \sum \{\lambda_i(\frac{1}{x_i}) : x_i \in F_\nu\}$ for $\nu = 1, 2, \dots, r$. We can eliminate the point \vec{z} from this system of equations by writing

$$(3.1) \quad \begin{cases} \sum \{\lambda_i(\frac{1}{x_i}) : x_i \in F_\nu\} = \sum \{\lambda_i(\frac{1}{x_i}) : x_i \in F_{\nu+1}\} \text{ for } \nu = 1, 2, \dots, r-1, \\ \sum \{\lambda_i : x_i \in F_1\} = 1. \end{cases}$$

Let us order the points \vec{x}_i within each block F_ν by increasing order of the index i , and the union $\cup_{\nu=1}^r F_\nu$ by letting F_μ precede F_ν whenever $\mu < \nu$. (In order to avoid duplication we could index the blocks F_ν by increasing order of the smallest index of their elements, i.e., $\mu < \nu$ iff $\min\{i : x_i \in F_\mu\} < \min\{i : x_i \in F_\nu\}$.) Denote by Λ the column of coefficients λ_i , ordered correspondingly. The equations (3.1) can be written as:

$$(3.2) \quad A \cdot \Lambda = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Where A is a square matrix of order $T(d, r) (= 1 + (d + 1)(r - 1))$ as illustrated below:

This system of (non-homogeneous) linear equations has a unique solution iff $\det A \neq 0$. Thus our polynomial is just $\det A$, regarded as a polynomial in the coordinates that appear as entries of A .

We still have to show that this polynomial is not identically 0. Let $\vec{e}_1, \dots, \vec{e}_d$ be the standard orthonormal basis of \mathbb{R}^d . Recall that $|F_\nu| = d + 1 - \varepsilon_\nu$ ($\nu = 1, 2, \dots, r$), where $1 \leq \varepsilon_\nu$ and $\sum_{\nu=1}^r \varepsilon_\nu = d$. For $\nu = 1, 2, \dots, r$,



Case II: If $\sum_{\nu=1}^r \varepsilon_{\nu} > d$ (i.e., if $\sum_{\nu=1}^r |F_{\nu}| < T(d, r)$) then $\cap_{\nu=1}^r \text{aff } F_{\nu} = \emptyset$.

Violation of the condition $\cap_{\nu=1}^r \text{aff } F_\nu = \emptyset$ means that the system (3.2) $A \cdot \Lambda = (1, 0, \dots, 0)^t$ does have a solution. This is equivalent to saying that the last column of A^+ is a linear combination of the first q columns. This implies that $\text{rank } A^+ \leq q$, which is equivalent to saying that all $(q+1) \times (q+1)$ submatrices of A^+ have zero determinant or (in view of the character of the last column of A^+) that all $q \times q$ submatrices of A that do not use the first row of A have zero determinant.

Denote by A^- the rectangular matrix (of order $(d+1)(r-1) \times q$) obtained by deleting the first row of A , and let the polynomial P be the sum of squares of all $q \times q$ subdeterminants of A^- . Violation of the condition $\cap_{\nu=1}^r \text{aff } F_\nu = \emptyset$ implies $P = 0$.

We still have to show that this polynomial P does not vanish identically. For any choice of vectors $\vec{x}_1, \dots, \vec{x}_t$ we have the quadruple equivalence: $P(\vec{x}_1, \dots, \vec{x}_t) = 0 \iff \text{rank } A^- < q \iff$ the columns of A^- are linearly dependent \iff the homogeneous system $A^- \cdot \Lambda = 0$ of linear equations has a nonzero solution.

To complete the proof, we shall describe a particular substitution of vectors in \mathbb{R}^d for the variable vectors $\vec{x}_1, \dots, \vec{x}_t$, that will lead to a system $A^- \cdot \Lambda = \vec{0}$ whose only solution is $\Lambda = \vec{0}$. Let $\vec{u}_0, \vec{u}_1, \dots, \vec{u}_d$ be vectors in \mathbb{R}^d whose only linear dependence (up to proportion) is $\sum_{i=0}^d \vec{u}_i = \vec{0}$. (Say, $\vec{u}_i = \vec{e}_i$ for $i = 1, \dots, d$ and $\vec{u}_0 = -\sum_{i=1}^d \vec{u}_i$. Define $U = \{\vec{u}_0, \vec{u}_1, \dots, \vec{u}_d\}$. For $\nu = 1, 2, \dots, r$ let E_ν be subsets of U that satisfy: $|E_\nu| = \varepsilon_\nu$ (where $|F_\nu| = d+1 - \varepsilon_\nu$), and $\cup_{\nu=1}^r E_\nu = U$. (Recall that $\sum_{\nu=1}^r \varepsilon_\nu \geq d+1$). Substitute for the variable vectors $\vec{x}_i \in F_\nu$ (bijectively) the vectors $\vec{u}_j \in U \setminus E_\nu$. (There is no need to substitute anything for variable vectors \vec{x}_i that are not in $\cup_{\nu=1}^r F_\nu$, since they do not appear in A .) Note also that this substitution does not yield a set of t points in SGP in \mathbb{R}^d : each point $\vec{u}_j \in U$ may appear up to $r-1$ times on the list $\vec{x}_1, \dots, \vec{x}_t$.

Now a solution Λ of the homogeneous system of equations $A^- \cdot \Lambda = \vec{0}$ yields a point $\vec{z} \in \mathbb{R}^d$ that has r representations

$$(3.3) \quad \vec{z} = \sum_{\vec{u}_j \in U \setminus E_\nu} \lambda_{j\nu} \cdot \vec{u}_j \quad \text{for } \nu = 1, \dots, r$$

where the sum of the coefficients is constant: $\sum_{\vec{u}_j \in U \setminus E_\nu} \lambda_{j\nu} = \sum_{\vec{u}_j \in U \setminus E_{\nu+1}} \lambda_{j\nu+1}$ for $\nu = 1, \dots, r-1$. The numbers $\lambda_{j\nu}$ ($\nu = 1, \dots, r$, $\vec{u}_j \in U \setminus E_\nu$) are the entries of the column vector Λ . If $\Lambda \neq \vec{0}$, then for some ν , and some $\vec{u}_h \in U \setminus E_\nu$, $\lambda_{h\nu} \neq 0$. But $\vec{u}_h \in U = \cup_{\mu=1}^r E_\mu$, and therefore $\vec{u}_h \in E_\mu$ for some $\mu \neq \nu$. Consider the two representations: $\vec{z} = \sum \{\lambda_{j\nu} \vec{u}_j : \vec{u}_j \in U \setminus E_\nu\} = \sum \{\lambda_{j\mu} \vec{u}_j : \vec{u}_j \in U \setminus E_\mu\}$. They are different: \vec{u}_h appears with a non-zero coefficient $\lambda_{h\nu}$ in the first one, but does not appear at all in the second one, since $\vec{u}_h \in E_\mu$. In both representations, the sum of coefficients is the same. But this is impossible: If $\vec{z} = \sum_{j=0}^d \zeta_j \vec{u}_j$, then the most general representation of \vec{z} as a linear combination of $\vec{u}_0, \vec{u}_1, \dots, \vec{u}_d$ is $\vec{z} = \sum_{j=0}^d (\zeta_j + \alpha) \vec{u}_j$, $\alpha \in \mathbb{R}$, so different representations necessarily differ in the sum of coefficients.

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